Synchronization of nonlinear systems with distinct parameters: Phase synchronization and metamorphosis

Hayato Fujigaki, Makoto Nishi, and Tokuzo Shimada
Department of Physics, Meiji University, Higashi Mita 1-1-1, Kawasaki, Kanagawa 214, Japan
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Rather than the synchronization between identical chaotic systems, the "phase synchronization" among two or many nonlinear systems with distinct nonlinear parameters is investigated. It is observed that the dynamics of globally coupled N_1 periodic and N_2 chaotic systems can be reduced to that between matrix-coupled chaotic and periodic systems as the result of a two-step realization of the synchronization among the former systems. The same situation holds even for the array of distinct systems with a nearest-neighbor interaction.

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The brain is a network of a vast number of chaotic neurons and for the pattern recognition the requisite condition is supposed to be the synchronization among neurons and the appearance of the macroscopic coherent mode in the network. For instance, Ott, Grebogi, and Yorke [1] have presented recently an intriguing view that the intelligence is supported by a chaotic switch. That is, chaos lives with infinitely many unstable periodic orbits and a periodic orbit transits to another most efficiently via the chaos. A nice demonstration of this possibility is given for a single unit chaos. Thus if the cells in the network synchronize we can regard them as one unit and the regularity of a few units may also be applied to the network. This possibility is the case which we study in this article.

There exist interesting investigations on the synchronization of chaotic cells [2–5]. In particular, Pecora and Carroll [2] found that two chaotic attractors with exactly the same (slightly different) nonlinear parameters synchronize perfectly (keeping tiny distance) when they are coupled together in a way that one of the dynamical variables of the master flow is substituted for the corresponding one of the slave flow. One may regard this shared variable as a common driving term and others (two subsystems) as under the influence of it via feedback. Their finding that even the chaotic flows synchronize triggered further studies including the synchronization under common noise terms [6]. As for the coupled map system Kaneko found that the globally coupled map lattice is endowed with remarkably rich clustering structure and that the positive-negative switch can be realized among the clusters by input [3].

We should note that all of the previous investigations are directed to the finding of the precise synchronization between identical units. In this article we address ourselves to the question of what the outcome would be if we couple together the same systems but with distinct nonlinear parameters. By constructing a simple coupling model based on the globally coupled map lattice we find an amazing phenomena that the systems flow synchronizing in the phase but with different sizes and/or positions. This phenomena may be termed as the *phase synchronization*. Actually in the brain what is important is presumably the phase locking between the neurons and is not the precise status of each. Thus our observation of the phase synchronization may have an important consequence to the activity of the brain.

For two systems we find that two nonlinear systems evolve in interaction in two new phase synchronizing flows in the phase space, the new pattern depending on the coupling, even if the set of parameters in one system is set in the chaotic regime and that in the other in the periodic regime. The rule of the dependence is in general quite simple; if the coupling favors the chaotic (periodic) system the new pattern is generally also chaotic (periodic). For large number of systems we prepare N systems in two categories; N_1 chaotic and N_2 periodic systems. We find that a rule like the above holds that the majority wins the minority. We also find that in a certain way the population ratio (N_1/N) can be related to the matrix parameter θ in the $N=2$ model. The implication of this relation is the main theme of this article.

Let us first describe our model for the simple case of $N=2$. We take two nonlinear systems with variables more than one. As an example we take the Lorenz systems and we treat them as discretized maps with the parameters for one system in the chaotic regime $(r_1=28, b_1=8/3, P_1=10)$ and for the other in the periodic regime $(r_2=270, b_2=8/3, P_2=10)$. At each time step the systems $(i=1,2)$ first evolve via the flow equations

$$\begin{aligned} x_i(t+\Delta t) &= x_i(t)+P_i(y_i-x_i)(t)\Delta t\ , \\ y_i(t+\Delta t) &= y_i(t)+(-x_iz_i+r_ix_i-y_i)(t)\Delta t\ , \\ z_i(t+\Delta t) &= z_i(t)+(x_iy_i-b_iz_i)(t)\Delta t\ . \end{aligned} \tag{1}$$

Then they interact each other by a simple matrix with two continuous parameters ϵ and θ in only one of the three variables x,y,z.

For instance, for the "x coupling,"

$$\begin{aligned} (1-\epsilon_2)x_1+\epsilon_2x_2 &\mapsto x_1'\ , \\ (1-\epsilon_1)x_2+\epsilon_1x_1 &\mapsto x_2'\ , \end{aligned} \tag{2}$$

with $\epsilon_1=\theta\epsilon$, $\epsilon_2=(1-\theta)\epsilon$ $(0\le\epsilon,\theta\le 1)$. The mechanism is that system 1 receives the effect of system 2 with a coupling constant ϵ_2 and vice versa. The systems evolve repeating this two-step process of map and interaction. Roughly the parameter ϵ gives the coupling strength and the parameter θ acts as a weight factor between the two systems. The sequence of map and interaction is the usual one in the coupled map analysis [3]. The difference is that we couple only one dimension to create a 'mean field' and the other subsystems are evolving under the influence of this mean field. We should add that in our model we define the driving term as a weighted mean between the x_1 and x_2 rather than $\dot{x}_1$ and $\dot{x}_2$. The subsystems evolve as nonautonomous flows (discretized in the time step Δt) and the mean field is calculated as in the usual map model. Thus our model is a multidimensional map model with coupling in one dimension. In this respect we are using the (discretized) Lorenz system as a typical sample of the multidimensional map. However, interestingly, the model reduces to a smooth flow at the limit $\Delta t\to 0$ which can be explained from the following argument. The interaction (2) serves to focus the motion of units to the mean field while the nonlinear evolution (1) acts in defocusing direction. Under a certain balance the systems fall into the attractor. This is just the same with the usual coupled map model but our model differs from them in that the subdimensions [e.g., y_1 and y_2 in (1)] have distinct nonlinear parameters (r_i). Near the onset of the attractor the driving variables (x_1 and x_2) come close to each other and the variations due to the interaction (2) become as small as the variation in (1). As the effect the orbits of our model become continuous flow. In fact we have checked that all the following results are unchanged for any choice of sufficiently small Δt (typically $\Delta t\approx 10^{-4}$).

The above matrix form facilitates a way to interpolate various important limits using the parameters ϵ and θ. For instance in the limit of $\epsilon=1$ and $\theta=0$ or 1 our model reproduces the original model of Pecora and Carroll but with a drastic extension that the nonlinear parameters for each system are set at completely distinct values.

To explain our analysis of N systems, let us briefly look at the analysis of the globally coupled map (GCM) by Kaneko [3]. In his model N identical maps first evolve as

$$x_i(n+1)=f(x_i(n))=1-a[x_i(n)]^2 \tag{3}$$

and then interact via a mean field with a coupling (ϵ),

$$(1-\epsilon)x_i+\frac{\epsilon}{N}\sum_j^N x_j\mapsto x_i' \ . \tag{4}$$

Under a certain balance of the nonlinearity (a) and the coherence (ϵ) these maps divide into two clusters and fall into two attractors moving with periodicity two as $(+-+-+\cdots)$ and $(-+-+-\cdots)$. Here the $+$ $(-)$ denotes that the value of the attractor is larger (smaller) than the unstable fixed point x^* of the map $[x^*=(\sqrt{1+4a}-1)/2a]$ at even n. In this two cluster regime the maps (4) reduce to

$$\begin{aligned}(1-\epsilon_-)x_+ +\epsilon_- x_-&\mapsto x'_+\\ (1-\epsilon_+)x_- +\epsilon_+ x_+&\mapsto x'_-\end{aligned} \tag{5}$$

and $\epsilon_\pm=[N_\pm/(N_+ +N_-)]\epsilon$, where N_+ (N_-) denotes the number of $+$ $(-)$ cells.

The main interest in our large N analysis is the possibility of phase synchronization among many systems with distinct parameters. As the simplest setup we take N_1 systems in the chaotic regime and N_2 systems in the periodic regime. After the evolution (1) in one time step Δt they interact by the same equation as (4) and the y and y' (z and z') do not interact directly. Suppose that the N_1 systems and the N_2 systems synchronize among each and change into two clusters. Then the complicated dynamics expressed by N coupled equations is expected to reduce to a more simple $N=2$ dynamics. This will allow us to repeat formally the reduction from (4) to (5) and relate the dynamics of (N_1,N_2) systems to that of the $N=2$ systems by a simple rule

$$\frac{N_1}{N_1+N_2}=\theta \tag{6}$$

and the subsequent phase synchronization between the two clusters will be observed. This should be checked by the observation that the synchronizing trajectories in the large N systems with certain population ratio N_1/N agree with those in $N=2$ case at the weight factor θ given by the rule (6). We will see in the following that the two cluster formation does take place and that the conjectured reduction holds perfectly in our model. Of course the dynamical requirements enabling the reduction of N to 2 are completely different between the GCM and our model. In the GCM the requirement is that *identical* systems organize themselves into two attractors. In our model the requirement is that N_1 systems and N_2 systems, which are *distinct* in the nonlinear parameters, fall into two clusters before the phase synchronization between the two. This crucial difference should not be overlooked. Now we are ready to present our results in order.

(1) $N=2$ *and the strongest* $(\epsilon=1)$ *and one-way* $[\theta=0$ or $1]$ *limit.* First we briefly review our results for $\epsilon=1$. The coupling for $\epsilon=1$ is

$$\begin{aligned}\theta x_1+(1-\theta)x_2&\mapsto x_1\\ (1-\theta)x_2+\theta x_1&\mapsto x_2\ .\end{aligned} \tag{7}$$

The limit $\epsilon=1$ and $\theta=0$ means $x_2\mapsto x_1$ and the limit $\epsilon=1$ and $\theta=1$ means $x_1\mapsto x_2$. Thus at $\epsilon_1=1$, $\theta=1$ system 1 becomes the master of the slave system 2. This is nothing but the case studied in the pioneering work by Pecora and Carroll who found that the two chaotic systems with (almost) the same parameters synchronize. We also analyze this extreme coupling case, but we set the two systems at completely different nonlinear parameters. What would the consequence be for the tight, one-way coupling between periodic and chaotic systems? Do they synchronize somehow or other? This question may look absurd since in the periodic system the phase is naturally defined while in the chaotic system only the trajectory length or diffeomorphic transform of it may be regarded as such. But we find that an amazing answer comes out. The slave system actually *metamorphose* into the same character with the master system and there motion is in complete *phase synchronization* with the master. For instance the periodic slave turned chaotic under the influence of the chaotic master and vice versa. This is by no means trivial, since the systems are coupled by only one of the dynamical variables and the other degrees of freedom are not in direct interaction. We present the figures of the metamorphoses in the following generic case.

(2) *The case for generic* ϵ $(0<\epsilon<1)$ *and both-way* θ $(0\leq\theta\leq1)$ *coupling.* We show in Fig. 1 the orbits of the coupled Lorenz systems for $\epsilon=0.3$. Figures 1(a) and 1(b) show the result of the coupling for $\theta=0.2$ and $\theta=0.8$, respectively. System 1 is set in the chaotic regime ($r_1=28$, $b_1=8/3$, $P_1=10$) and system 2 is in the periodic regime ($r_2=270$, $b_2=8/3$, $P_2=10$). To be explicit the coupling matrix(2) for the x variable at $\theta=0.2$ is

$$0.76x_1+0.24x_2\mapsto x_1\ , \qquad 0.06x_1+0.94x_2\mapsto x_2 \tag{8}$$

and at $\theta=0.8$

$$0.94x_1+0.06x_2\mapsto x_1\ , \qquad 0.24x_1+0.76x_2\to x_2\ . \tag{9}$$

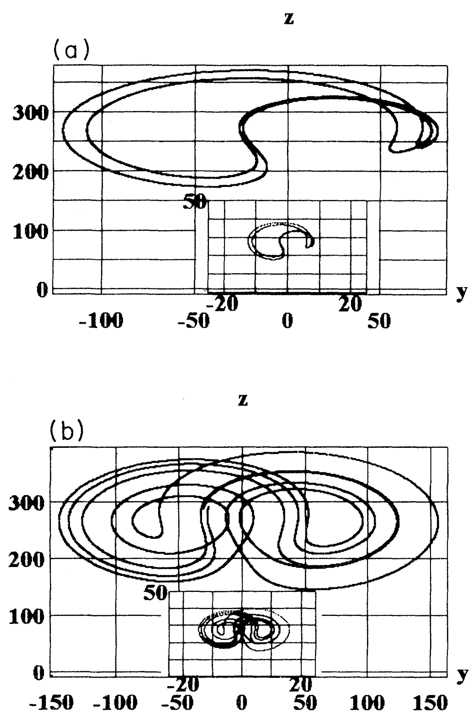

FIG. 1. The phase synchronizing orbits of x-coupled Lorenz systems in the y-z plane. $\epsilon=0.3$ and $\Delta t=10^{-4}$. System 1 in the chaotic regime ($r_1=28, b_1=8/3, P_1=10$) and system 2 in the periodic regime ($r_2=270, b_2=8/3, P_2=10$). Two systems have turned into mutually organized new shapes. (a) The coupling parameter $\theta=0.2$ [See Eq. (8)]. System 1 (chaotic) has metamorphosed under the effect of system 2 (periodic). Note the difference in the scale for each orbit. (b) The coupling parameter $\theta=0.8$. System 2 (periodic) has followed the system 1 (chaotic).

Metamorphosis in this case is in the way that both systems turned into mutually organized new shapes. In the small θ case [Fig. 1(a)] we find that the shape of the periodic system (system 2) is not affected much while the chaotic system (system 1) metamorphoses into periodic flow in due course of interaction. Thus, for small θ, system 2 wins in the competition of the structure decision. In contrast for large θ the winner is system 1 [see Fig. 1(b)]. The synchronization is so perfect that we do not need to show the Lissajous plots. The Lissajous contour is simply a diagonal line of a rectangle for any initial points in the basin. We also have checked that the phase synchronization occurs independently from the choice of the coupling variables as far as the sub-Lyapunov exponent is not positive [2]. However, there seems no unique way to compare the resulting two orbits. For instance, in the case of the x-coupled Lorenz models depicted in Fig. 1, we need two scaling factors in both the y and z directions in order to compare the two orbits. On the other hand in the y-coupled Lorenz system the orbits turn out with almost the same size. Furthermore, in the case of the Rössler model there is no sizable difference but some parallel shift makes the two trajectories almost overlap. Despite these differences we find that in every case the phase synchronization is perfect.

The most important question now is why the parameter θ determines which is the winner among the chaotic and the periodic systems. At the limit $\epsilon=1$, this can be naturally understood because as we saw above $\theta=1$ (0) means system 1 is the master (slave) and the θ smoothly interpolates between these limits. However, at the generic ϵ the case is more subtle as is seen in the nontrivial form of the coupling matrix. For instance at $\theta=0.2$ the first equation of (8) dictates that system 1 wins the game while the second equation dictates the other way round and one cannot tell which is the fate of the flows. We will obtain a plausible answer in the analysis of the N coupled systems.

(3) *The Poincaré map and Lyapunov exponents.* The role of the weight θ is best illustrated by the Poincaré map and the Lyapunov exponents in Figs. 2(a), and 2(b), respectively. The Poincaré map is evaluated by the cut of the contours with the conditions $\dot{z}_i=0$ and $\ddot{z}_i<0$ $(i=1,2)$.

The Lyapunov exponents are evaluated by the basic method [7] by keeping track of the expanding rate of volume of the parallelepiped in the six dimensional phase space. The θ is varied from 0 to 1 in step 0.002 and at any θ the systems remain in phase synchronization. Figure 2 shows that in the small θ region the systems synchronize in periodic flows and in the large θ region in chaotic flows. The parameter θ indeed acts as a novel control parameter of the nonlinearity of the whole two systems. There are clear periodic windows among the chaos and the agreement in the position of the windows in Figs. 2(a) and 2(b) is perfect.

(4) *The adiabatic change from period down to chaos.* It is interesting to test the ability of the parameter θ to control the nonlinearity of the phase synchronizing systems. By ability we mean that we can rapidly change θ (for instance from $\theta=0.2$ to $\theta=0.8$ within only ten turns of the orbits) keeping perfect phase synchronization between the systems. It is fun to press the up and down keys for the θ parameter and watch the dance of the synchronizing orbits in the display. The phase synchronization is quite tight. In Fig. 3 we show some example of this. The parameters of the models are the same as those for Fig. 2. We adopt the y coupling and show the x-z plot. Just for the purpose of illustration the θ is varied continuously with the time t by $\theta=2/\pi \tan^{-1}(t)$.

(5) *The N-coupled systems.* Our model consists of N-coupled Lorenz systems: the N_1 systems with parameters in the chaotic regime and the N_2 systems in the periodic regime and they evolve from completely random starting points.

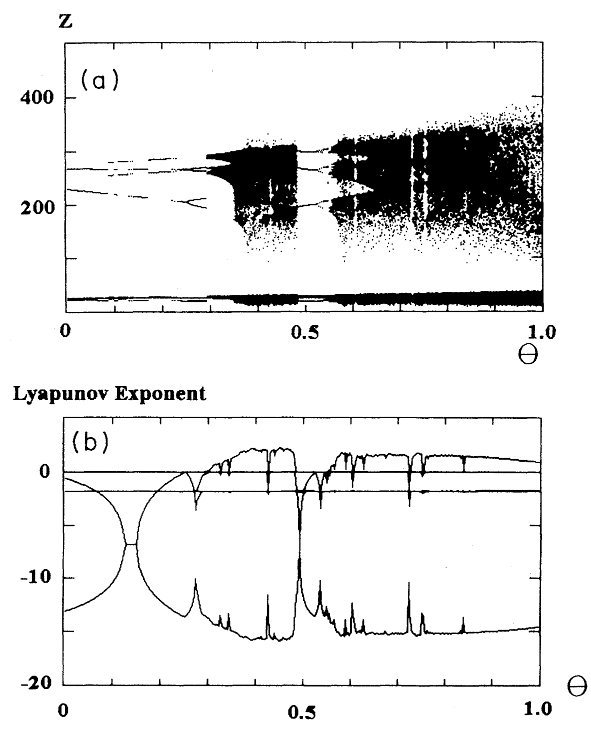

FIG. 2. The same Lorenz systems as Fig. 1 (but $r_2=300$). For small θ the two systems synchronize in periodic flows and for large θ in chaotic flows. The agreement in the position of the windows is perfect. (a) The Poincaré map versus θ with the cut conditions $\dot{z}_i=0$ and $\ddot{z}_i<0$ $(i=1,2)$. (b) The six Lyapunov exponents versus θ. One exponent is at zero and the line around $\lambda=-2$ is a twofold degenerate. The minimum exponent moves far down from the frame.

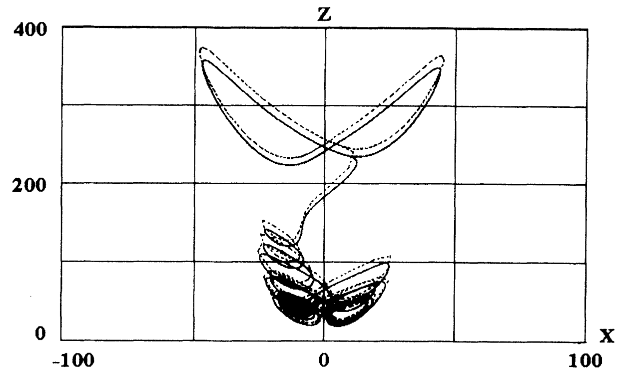

FIG. 3. The adiabatic change of y-coupled flows in the x-z plane in perfect phase synchronization from period down to chaotic regime. The parameters of the models are the same as those for Fig. 2. The solid curve for system 1 and the dashed curve for system 2. The $\theta=2/\pi \tan^{-1}(t-t_0)$ and t_0 is the time that the flows stabilized in the periodic orbits.

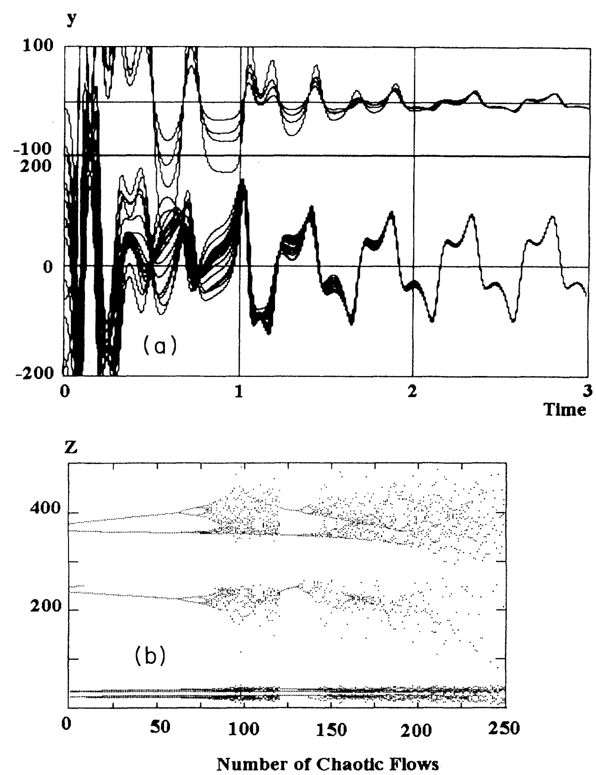

FIG. 4. The N Lorenz system globally coupled in the variable x_i; N_1 chaotic systems $(r=28, b=8/3, P=10)$ and the N_2 periodic systems $(r=270, b=8/3, P=10)$. Random start. (a) The two-step process to the perfect phase synchronization. Parameters are the same as Fig. 1. The $N_1=4$ and $N_2=16$ systems gradually bunch together among each during $t=0$ to $t=2$ and the two bunches mutually derives themselves into phase synchronization after $t=2$. (b) The Poincaré map of totally $N=250$ Lorenz systems versus the number of chaotic members N_1. This agrees with the $N=2$ Poincaré map in Fig. 2(a), especially as for the pattern of the periodic windows structure and gives an additional verification of the two-step dynamics via the relation [see Eq. (6)].

Globally coupling. First let us investigate the mean field type coupling (4) and the y_i and y_i' (z_i and z_i') do not interact directly. We observe that the initial N trajectories soon change into two perfectly phase synchronizing orbits which look just like those in Fig. 1. As is clearly seen in Fig. 4(a) the synchronization proceeds in two steps. First each of the N_1 and N_2 systems bunch together among each. Then the two bunches mutually interact—just like the two units in the $N=2$ analysis—and finally the whole N systems derive themselves into two phase synchronizing orbits in the phase space, one consist of N_1 units and the other of N_2 units. As we have discussed at (6) this two-step process should also be checked by the correspondence between the population ratio in the N systems and the weight factor θ in the $N=2$ matrix coupling model (2). Figure 4(b) exhibits the Poincaré map of the globally coupled N_1 chaotic and N_2 periodic systems with respect to the number of chaotic systems (N_1). The agreement with the $N=2$ Poincaré map in Fig. 2(a)—even as for the pattern of the periodic windows structure— gives further verification of this two-step dynamics via the relation (6).

In the $N=2$ analysis we wished to seek out the real reason of the tendency that the large θ favors system 1. As relation (6) is now established, the tendency that the large (small) θ favors system 1 (2) can be rephrased in terms of the N systems that the majority wins over the minority systems. This gives a plausible answer to the question posed in the $N=2$ analysis.

N systems with nearest-neighbor couplings. In the globally coupled model there is not notion of the distance. By this simplification the essential feature of the synchronization can be best studied in a scale invariant manner but the lack of the distance forbids the analysis of the spatiotemporal structure of the synchronization. Therefore, we set N_1 (chaotic) and N_2 (periodic) systems in random combinations on a circle and let them interact with the nearest-neighbor both-way coupling

$$\left[1-\frac{\epsilon}{2}\right]x_i+\frac{\epsilon}{2}x_{i+1}\mapsto x_i\ ,$$
$$\left[1-\frac{\epsilon}{2}\right]x_{i+1}+\frac{\epsilon}{2}x_i\mapsto x_{i+1}\quad (i=1,\ldots,N)\ . \tag{10}$$

Figure 5 shows the spatiotemporal plot of the evolution of the $N=20$ systems with $N_1=16$ and $N_2=4$ from the random initial values and $\epsilon=0.3$. Despite that the four periodic systems are far separated from each other by the chaotic systems, we see that amazingly they quickly turn into mutual synchronization as is observed as synchronizing peaks. This "barrier penetration" also occurs for the other 16 chaotic systems and the chaotically oscillating sheet exhibits their synchronizing motion. This first step—the cluster formation—completes roughly in 2 sec and then, as the second step, the periodic cluster metamorphose into perfect phase synchronization with the chaotic cluster. For illustration we can only show the small N case but we have numerically confirmed this route to the synchronization in larger systems and also in the random spatial distribution of the N_1 and N_2 systems. Thus the phase synchronization in the nearest-neighbor model also proceeds in two steps.

In conclusion we investigated the coupled distinct nonlinear systems. We found that they derive themselves into *phase synchronization.* For $N=2$ we constructed a simple matrix model and found that we can control the synchronization by a single parameter in the model θ. For large N we considered the coupling of N_1 systems with parameters in the chaotic regime and N_2 systems in the periodic regime. We again found a complete phase synchronization. Furthermore, we found that the synchronization in many systems proceeds in two steps; the like systems first synchronize among themselves forming

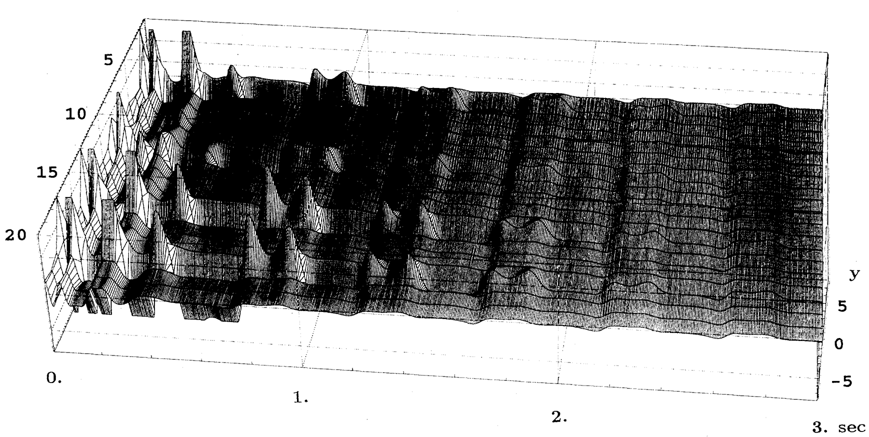

FIG. 5. The $N=20$ systems with nearest-neighbor both-way ($\theta=0.5$) x couplings ($\epsilon=0.3$) and the spatiotemporal structure of the two-step process to the phase synchronization. $N_1=16$ (chaotic) and $N_2=4$ (periodic) systems. The y coordinates of the latter are scaled down by factor 20. The four periodic systems are at $i=2$, 8, 13, 17 and are far separated from each other by the other 16 chaotic systems. The first process completes in 2 sec; the periodic systems quickly turn into four synchronizing peaks due to penetrating interaction across the chaotic systems and the chaotic systems form a coherently oscillating sheet. In the second process, the periodic peaks metamorphose into perfect phase synchronization with the chaotic cluster.

two clusters and then as two units they start the metamorphosis to the final phase synchronization. We verified this two-step process by the direct inspection as well as by the check of the validity of the relation (6) between the dynamics of the N systems and the two systems.

We often come across the case that the synchronization in the phase is the important issue while the magnitudes of components are not much relevant. For instance in the path integral the classical trivial and nontrivial configurations dominate the amplitude as all nearby paths contributes coherently with different magnitudes. We have relaxed the notion of synchronization to the phase synchronization and in this freedom we have seen an interesting possibility of a reduction of the large system dynamics to the small system dynamics.

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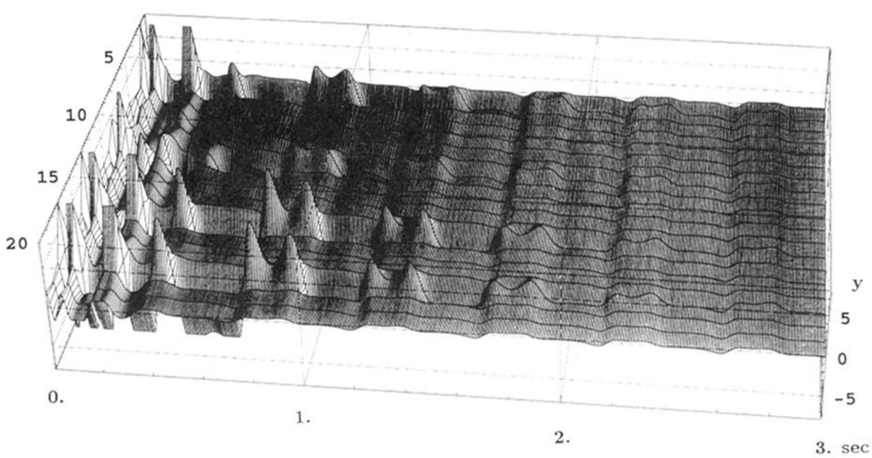

FIG. 5. The $N = 20$ systems with nearest-neighbor both-way ($\theta = 0.5$) x couplings ($\epsilon = 0.3$) and the spatiotemporal structure of the two-step process to the phase synchronization. $N_1 = 16$ (chaotic) and $N_2 = 4$ (periodic) systems. The y coordinates of the latter are scaled down by factor 20. The four periodic systems are at $i = 2$, 8, 13, 17 and are far separated from each other by the other 16 chaotic systems. The first process completes in 2 sec; the periodic systems quickly turn into four synchronizing peaks due to penetrating interaction across the chaotic systems and the chaotic systems form a coherently oscillating sheet. In the second process, the periodic peaks metamorphose into perfect phase synchronization with the chaotic cluster.